Bright solitons on a continuous wave background for the inhomogeneous nonlinear Schrödinger equation in plasma

2006 J. Phys. A: Math. Gen. 39 11947

(http://iopscience.iop.org/0305-4470/39/38/019)

J. Phys. A: Math. Gen. **39** (2006) 11947–11953 doi:10.1088/0305-4470/39/38/019

Bright solitons on a continuous wave background for the inhomogeneous nonlinear Schrödinger equation in plasma

Lei Wu[1], **Qin Yang**[2] **and Jie-fang Zhang**[3,1]

[1] Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua 321004, People's Republic of China
[2] School of Electrical and Mechanical Engineering, Jiaxing University, Jiaxing 314001, Zhejiang, People's Republic of China
[3] CCAST(World Lab.), PO Box 8730, Beijing 10080, People's Republic of China

E-mail: jf_zhang@mail.zjnu.net.cn

Received 25 June 2006, in final form 15 August 2006
Published 5 September 2006
Online at stacks.iop.org/JPhysA/39/11947

Abstract
In this paper, the inhomogeneous damped nonlinear Schrödinger equation (NLS) which describes wave propagation in plasmas is investigated. Based on Husimi's transformation and lens-type transformation, we reduce the inhomogeneous NLS equation to the standard NLS equation and thus the bright soliton solutions on a continuous wave (cw) background are constructed and discussed. Besides, we also consider the inhomogeneous NLS equation with different damping coefficients. Based on the perturbation theory for bright solitons, the approximate bright soliton solutions for such an equation are obtained, which are in good agreements with direct numerical simulations.

PACS numbers: 02.30.Jr, 05.45.Yv, 52.35.Mw

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The solution of the nonlinear Schrödinger equation in an inhomogeneous medium is of great importance for investigating wave propagation in various types of physical situations such as plasma physics, nonlinear optics, condensed matter [1–3] and so on. It is clear that wave propagation in plasmas is strongly affected by both the nonlinearity and nonuniformity of the medium, and it is interesting to ask whether the nonlinear density profile modification can significantly change the wave-packet propagation in an inhomogeneous plasma, or more fundamentally, whether the inhomogeneous plasma still supports solitons. To address these questions, we present the analytic solutions of the NLS equation in a parabolic inhomogeneous plasma with collisional damping. The propagation of a very large wavelength, with a small but finite amplitude electron plasma wave packet in a medium with a parabolic density and constant collisional damping is governed by the following inhomogeneous NLS equation [4]:

$$\mathrm{i}q_t + q_{xx} + 2|q|^2 q - (\alpha x - \beta^2 x^2)q + \mathrm{i}\beta q = 0, \tag{1}$$

where αx, $\beta^2 x^2$ correspond to linear and parabolic density profiles, respectively, and the last term $\mathrm{i}\beta q$ corresponds to damping ($\beta > 0$ is the damping coefficient). Equation (1) may serve as a model for the tunnelling of solitons across density humps in a loss medium. The one-soliton solutions of equation (1) were given through a reduction method [4] and two-soliton solutions were given through the inverse scattering transform [5].

It is worth noting that, for the condition $\beta = 0$, equation (1) reduces to

$$\mathrm{i}q_t + q_{xx} + (2|q|^2 - \alpha x)q = 0, \tag{2}$$

which can be used to model Langmuir or electromagnetic wave propagation with a linear time-independent density [6] and quasi-one-dimensional BEC in a time-dependent linear external potential [8]. Equation (2) has been discussed in some papers [6, 7]. They employed Husimi's transformation to change the space variable from x to x' where $x' = x - \xi(t)$. This x' is the coordinate with respect to the moving origin $\xi(t)$. Later, $\xi(t)$ will be taken as the centre of mass of the soliton and equation (2) can be reduced to the well-known standard NLS equation. Then the N-soliton solution can be easily constructed.

For the condition $\alpha = 0$, equation (1) reduces to

$$\mathrm{i}q_t + q_{xx} + (2|q|^2 + \beta^2 x^2)q + \mathrm{i}\beta q = 0. \tag{3}$$

In this case, the quantity $\int_{\infty}^{\infty} |q(x,t)|^2\, \mathrm{d}x$ will decay exponentially as $\mathrm{e}^{-2\beta t}$ due to the collisional damping. One may employ a lens-type transformation [9, 10] to eliminate the x^2 term and reduces equation (3) to the NLS equation without explicit spatial dependence.

In this paper, based on Husimi's transformation and lens-type transformation, we reduce equation (1) to the standard NLS equation. Then we easily get the soliton solutions for equation (1) because the standard NLS equation is well studied and many solutions have been obtained. What is more, we also investigate the inhomogeneous NLS equation with different damping coefficients based on the perturbation theory for bright solitons.

2. Reduction to the standard NLS equation

In this section, we use Husimi's transformation and lens-type transformation to reduce equation (1) to the standard NLS equation. To obtain the bright soliton solutions for equation (1), we use the following ansatz

$$q(x,t) = \ell^{-1} \exp\left[\mathrm{i}\left(gx^2 + fx + \int_0^t f^2\, \mathrm{d}t\right)\right] u(X,T), \tag{4}$$

where f, g are real functions of time t, $X = \frac{x-\xi(t)}{\ell(t)}$ and $T = T(t)$. In fact, when $\xi(t) = 0$ and $f = 0$, the ansatz is just the lens-type transformation, while for $\ell(t) = 1$ and $g = 0$ the ansatz is Husimi's transformation.

By demanding that

$$g_t + 4g^2 - \beta^2 = 0, \tag{5}$$

$$f_t + 4fg + \alpha = 0, \tag{6}$$

$$l_t - 4gl = 0, \tag{7}$$

$$T_t - \ell^{-2} = 0, \tag{8}$$

$$\xi_t - 2f - 4g\xi = 0, \tag{9}$$

we convert equation (1) to

$$\mathrm{i}u_T + u_{XX} + 2|u|^2u = \mathrm{i}\epsilon(t)u, \tag{10}$$

where $\epsilon = (2g - \beta)\ell^2$.

From the expression for ϵ, we can clearly see that if $\beta = 2g$, equation (10) can be reduced to the standard NLS equation. That is to say, equation (1) is integrable when $\beta = 2g$. Besides, we can also conclude that the linear density profile α does not affect the integrability of equation (1), so α can be time dependent. But β must be a constant, because if β is time dependant, then from equation (5) we have $\beta \neq 2g$, then we cannot retrieve the standard NLS equation. In the following, we view β as a constant and $g = \frac{\beta}{2}$; thus we retrieve the standard NLS equation

$$\mathrm{i}u_T + u_{XX} + 2|u|^2u = 0, \tag{11}$$

which has been well studied and various theoretical solutions have been obtained, including the Jacobian elliptic function solutions, N-bright soliton solutions and bright soliton solutions propagate on a cw background [11]. In this paper, we only focus on the bright one-soliton solution

$$u(X, T) = \eta\,\mathrm{sech}[\eta(X - 2vT)]\exp[\mathrm{i}(vX + (\eta^2 - v^2)T)] \tag{12}$$

and the bright soliton solution propagates on a cw background

$$u(X, T) = \exp(2\mathrm{i}a^2T)\left[a + b\frac{b\cos(cT) + \mathrm{i}\nu\sin(cT)}{\nu\cosh(bX) + 2a\cos(cT)}\right], \tag{13}$$

where η, v, b are arbitrary constants, a is the cw background and $c = b\nu = b\sqrt{4a^2 + b^2}$. It should be noted that, if we take the limit $a \to 0$, solution (13) approaches to the bright soliton. Then we regard this solution as the general type of bright soliton.

3. Bright soliton on a cw background

From the analysis in section 2, we have $\beta = 2g$. By solving equation (6)–(9), we have $\ell = \mathrm{e}^{2\beta t}$, $T = \frac{1-\exp(-4\beta t)}{4\beta}$, $f = \left(-\int_0^t \alpha\,\mathrm{e}^{2\beta t}\,\mathrm{d}t + C_1\right)\mathrm{e}^{-2\beta t}$, $\xi = \int_0^t \left(-2\int_0^t \alpha\,\mathrm{e}^{2\beta t}\,\mathrm{d}t + 2C_1\right)\mathrm{e}^{-4\beta t}\,\mathrm{d}t + C_2]\,\mathrm{e}^{2\beta t}$. Thus the exact bright soliton solution on a cw background for equation (1) can be easily derived by combining these solutions with the ansatz (4) and solution (13).

In order to understand the influence of the parameter α and β on the dynamics of bright soliton on a cw background, we consider the linear density profile and parabolic density profile separately in the following two cases.

3.1. Linear density cases

In this case, $\beta = 0$, so $\ell = 1$, $T = t$, $f = -\int_0^t \alpha(t)\,\mathrm{d}t + C_1$, $\xi = -2\int_0^t\int_0^t \alpha(t')\,\mathrm{d}t'\,\mathrm{d}t + 2C_1 t$ $+ C_2$. The bright soliton solution on a cw background is

$$q(x, t) = \left|a + b\frac{b\cos(ct) + \mathrm{i}\nu\sin(ct)}{\nu\cosh\left[b\left(x + 2\int_0^t\int_0^t \alpha(t')\,\mathrm{d}t'\mathrm{d}t - 2C_1t - C_2\right)\right] + 2a\cos(ct)}\right|. \tag{14}$$

Solution (14) shows that, when compared with solitary wave solution of the standard NLS equation, the main characteristics of the solution in the presence of linear density profile $\alpha(t)$ are essentially the same except for the change of soliton velocities. The soliton velocity now

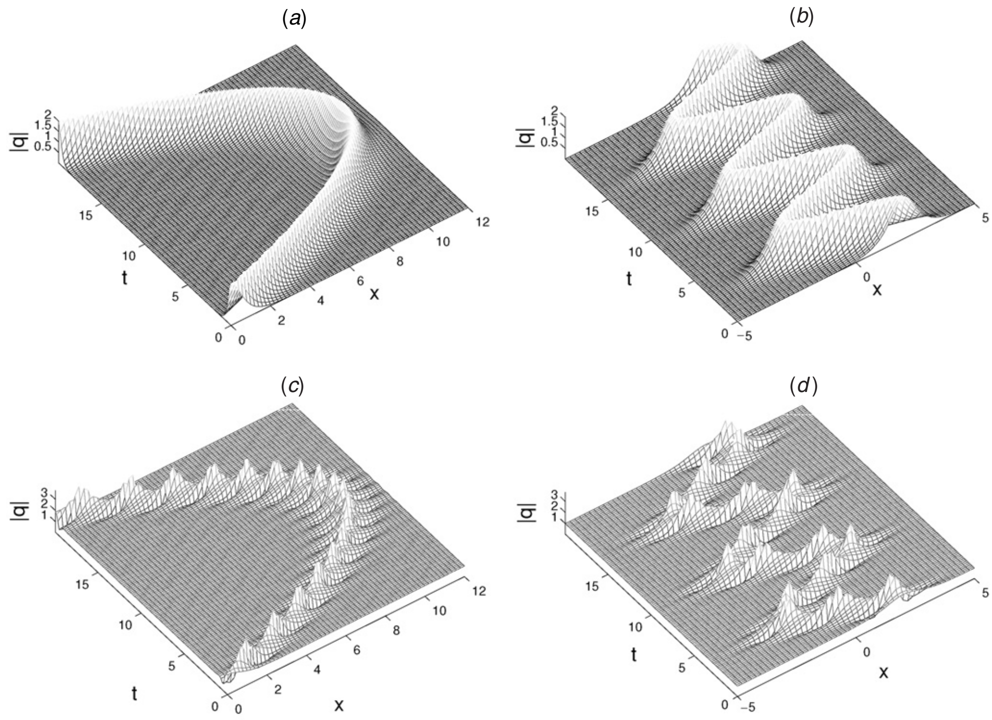

Figure 1. Dynamics of bright solitons on a cw background in the presence of $\alpha(t)$ when $b = 2$: (*a*) $a = 0, \alpha = 0.1, C_1 = 1, C_2 = 0$, (*b*) $a = 0, \alpha = \cos(t), C_1 = 0, C_2 = 2$, (*c*) $a = 1, \alpha = 0.1, C_1 = 1, C_2 = 0$, (*d*) $a = 1, \alpha = \cos(t), C_1 = 0, C_2 = 2$.

is $\dot{x} = 2C_1 - 2\int_0^t \alpha(t')\,\mathrm{d}t'$. So if α is a constant, the trajectory of bright soliton will be a parabola (figure 1(*a*) and (*c*)), while for $\alpha = \cos(\omega t)$, the trajectory of bright soliton will become to oscillate with the period $\frac{2\pi}{\omega}$ (figure 1(*b*) and (*d*)).

It is also interesting to note that

$$\int_{-\infty}^{\infty} [|q(x,t)|^2 - |q(\pm\infty,t)|^2]\,\mathrm{d}x = 2b, \tag{15}$$

which is the exact energy of electron plasma wave in the bright soliton against the cw background. This indicates that the energy of bright soliton remains invariant in the presence of linear density profile α. In contrast, the quantity

$$\int_{-\infty}^{\infty} |q(x,t) - q(\pm\infty,t)|^2\,\mathrm{d}x = 2b + abM\cos(ct), \tag{16}$$

where

$$M = \frac{4\mathrm{arc\,tan}\sqrt{\frac{\nu^2 + a\cos(ct)}{\nu^2 - a\cos(ct)}}}{\nu^2 - a^2\cos^2(ct)}, \tag{17}$$

shows that a time-periodic energy change is performed between the bright soliton and the cw background. From this solution we can see that, in the case of zero background, there will be no exchange of energy. However, in the case of a nonzero cw background, the exchange of energy between the bright soliton and the cw background becomes quicker as c increases, as shown in figure 2(*a*).

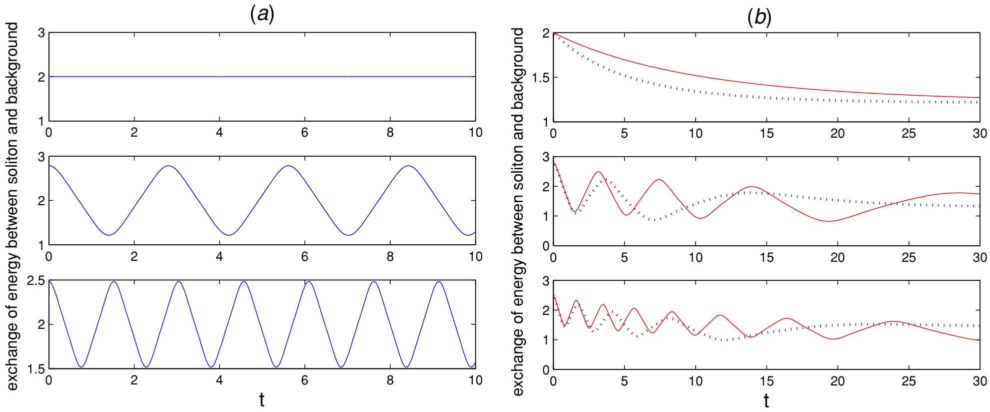

Figure 2. Energy change between the bright soliton and the background when $b = 1$ and from the top panel to the bottom the parameter a corresponds to 0, 1, 2: (*a*) the linear density profile, (*b*) parabolic density profile. The solid lines correspond to $\beta = 0.02$ while dotted lines correspond to $\beta = 0.04$.

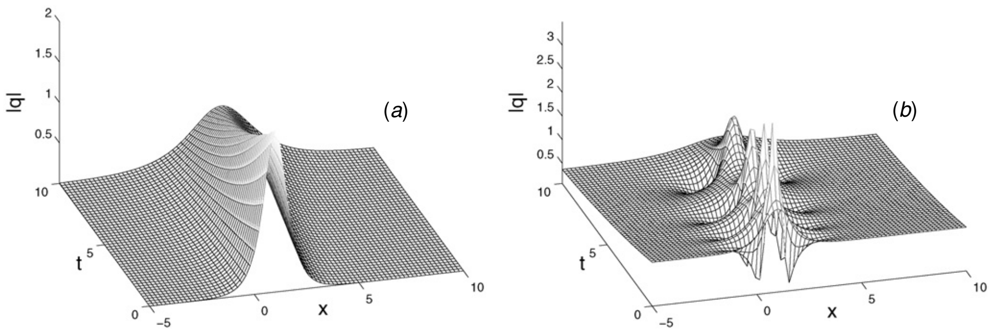

Figure 3. Dynamics of bright solitons on a cw background in the presence of β when $b = 2$, $\beta = 0.05$, $C_2 = 1$: (*a*) $a = 0$, (*b*) $a = 1$.

3.2. Parabolic density cases

In this case, $\alpha = 0$. So $\ell = \mathrm{e}^{2\beta t}$, $T = \frac{1-\exp(-4\beta t)}{4\beta}$. For simplicity we choose $f = 0$, so $\xi = C_2\,\mathrm{e}^{2\beta t}$. The bright soliton solution on a cw background is

$$q(x,t) = \mathrm{e}^{-2\beta t}\left| a + b\frac{b\cos\left(c\frac{1-\exp(-4\beta t)}{4\beta}\right) + \mathrm{i}\nu\sin\left(c\frac{1-\exp(-4\beta t)}{4\beta}\right)}{\nu\cosh[b(x\mathrm{e}^{-2\beta t} - C_2)] + 2a\cos\left(c\frac{1-\exp(-4\beta t)}{4\beta}\right)}\right|. \qquad (18)$$

From solution (18) we know that the parabolic density profile and damping term not only change the soliton amplitude and cw background (as shown in figure 3) but also change the soliton velocity. The soliton's velocity is now $\dot{x} = 2\beta C_2\,\mathrm{e}^{2\beta t}$ which means the soliton accelerates in the nonuniform media. In addition, as opposed to the linear density profile, the energy of the bright soliton in a parabolic density profile decays due to the damping term $\mathrm{i}\beta q$:

$$\int_{-\infty}^{\infty} [|q(x,t)|^2 - |q(\pm\infty,t)|^2]\,\mathrm{d}x = 2b\,\mathrm{e}^{-2\beta t} \qquad (19)$$

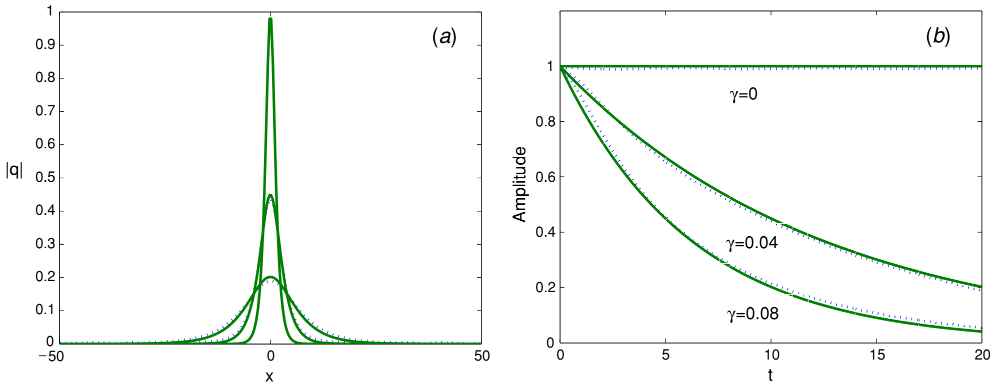

Figure 4. Comparison between the theoretical solution (solid line) and direct numerical solution (dotted line) when $\beta = 0.05$, (*a*) $\gamma = 0.04$, the three curves, from top to bottom, correspond to $t = 0$, $t = 10$, $t = 20$, (*b*) evolution of bright soliton amplitude for different values of γ.

and the exchange between the bright soliton and the cw background becomes

$$\int_{-\infty}^{\infty} |q(x,t) - q(\pm\infty,t)|^2 \, \mathrm{d}x = [2b + abM' \cos(ct)] \, \mathrm{e}^{-2\beta t}, \tag{20}$$

where M' can be obtained by replacing t by $\frac{1-\exp(-4\beta t)}{4\beta}$ in equation (17). In the presence of the damping term, the exchange of energy between the soliton and the cw background acts almost the same as the linear density case, except the exchange amplitude decays due to the collisional damping. As shown in figure 2(*b*), in the case of zero background, there will be no exchange of energy, but the soliton energy will decay due to the damping term. However, in the case of nonzero cw background, the exchange of energy between the bright soliton and the cw background becomes quicker as c increases provided β remains constant. As for the same value of c, the exchange becomes slower as β increases.

4. Inhomogeneous NLS equation with damping term iγq

In this section, as an extension to the above analysis, we consider the following inhomogeneous NLS equation in a more general form:

$$\mathrm{i}q_t + q_{xx} + 2|q|^2 q + \beta^2 x^2 q + \mathrm{i}\gamma q = 0. \tag{21}$$

Following the above analysis, we set the ansatz $q(x,t) = \ell^{-1}\exp(\mathrm{i}gx^2)u(X,T)$ to reduce equation (21) to

$$\mathrm{i}u_T + u_{XX} + 2|u|^2 u = \mathrm{i}(2g - \gamma)\ell^2, \tag{22}$$

where $g = \frac{\beta}{2}$, $\ell = \mathrm{e}^{2\beta t}$, $X = x\,\mathrm{e}^{-2\beta t}$ and $T = \frac{1-\exp(-4\beta t)}{4\beta}$.

Obviously, for $\gamma \neq \beta$, equation (21) is not integrable. But when $\beta \ll 1$, $\gamma \ll 1$, we can take the right-hand side of equation (22) as a small perturbation. When this perturbation is absent, equation (22) has a bright soliton solution (12). Then, according to the perturbation theory for the bright soliton [12, 13], we know that the amplitude of the bright soliton of equation (22) decays at a rate $\exp\left[\int_0^T 2(\beta - \gamma)\ell^2 \, \mathrm{d}T\right] = \exp\left[\int_0^t 2(\beta - \gamma) \, \mathrm{d}t\right]$, while the width increases at a rate $\exp\left[\int_0^t 2(\gamma - \beta) \, \mathrm{d}t\right]$. So equation (21) has an approximate bright soliton solution

$$|q(x,t)| = \eta \, \mathrm{e}^{-2\gamma t} \mathrm{sech}[\eta \, \mathrm{e}^{-2\gamma t} x]. \tag{23}$$

In figure 4 we compare the time evolution of the approximate solution (23) to that of the direct numerical solution of equation (21). The initial condition was chosen as $q(x,0) = \operatorname{sech}(x)\exp\left(\mathrm{i}\frac{\beta}{2}x^2\right)$. Figure 4(*a*) shows the evolution of a bright soliton with damping coefficient $\gamma = 0.04$. It should be emphasized that the approximate solution reveals excellent agreement with the numerical solution. The soliton shape seems well preserved in time and its amplitude decreases while its width increases. The time evolution of the soliton amplitude for different values of γ is shown in figure 4(*b*). We can see even when γ increases, the approximate solution also agrees quite well with the numerical one.

5. Conclusion

In conclusion, we have considered the damped and inhomogeneous NLS equation. First, we reduced the inhomogeneous NLS equation to the standard NLS equation by employing Husimi's transformation and lens-type transformation, and then constructed the bright soliton solution on a cw background. Especially, when the cw background approaches to zero, we get a one-bright soliton solution. It should be noticed that, the transformation presented in this paper can also be used to reduce the coupled inhomogeneous NLS equations:

$$\mathrm{i}q_{1t} + q_{1xx} + 2(|q_1|^2 + |q_2|^2)q_1 - (\alpha x - \beta^2 x^2)q_1 = 0,$$
$$\mathrm{i}q_{2t} + q_{2xx} + 2(|q_1|^2 + |q_2|^2)q_2 - (\alpha x - \beta^2 x^2)q_2 = 0$$

to the coupled homogeneous NLS equation:

$$\mathrm{i}u_{1T} + u_{1XX} + 2(|u_1|^2 + |u_2|^2)u_1 = 0, \qquad \mathrm{i}u_{2T} + u_{2XX} + 2(|u_1|^2 + |u_2|^2)u_2 = 0$$

by the transformations $q_i(x,t) = \ell^{-1}\exp\left[\mathrm{i}\left(gx^2 + fx + \int_0^t f^2\,\mathrm{d}t\right)\right]u_i(X,T), i = 1, 2.$

Second, we have considered the inhomogeneous NLS equation with the damping term $\mathrm{i}\gamma q$. This equation can be viewed as a more general form of inhomogeneous NLS equation. We have obtained an approximate bright soliton solution for such an equation and compared the approximate solution with the exact results derived by numerical simulation. As shown in figure 4, the approximate solution seems to reveal a satisfactory agreement with the numerical solution.

References

[1] Christiansen P L, Eilbeck J C and Parmentier R D (ed) 1993 *Future Directions of Nonlinear Dynamics in Physical and Biological Systems* (New York: Plenum)

[2] Sulem C and Sulem P L 1999 The nonlinear Schrödinger equation *Self-Focusing and Wave Collapse* (Springer: New York)

[3] Hasegawa A and Matsumoto M 2003 *Optical Solitons in Fiber* (Berlin: Springer)

[4] Gupta M R, Som B K and Dasgupta B 1978 *Phys. Lett.* A **69** 172

[5] Gupta M R 1979 *Phys. Lett.* A **72** 420

[6] Chen H H and Liu C S 1976 *Phys. Rev. Lett.* **37** 693

[7] Nogami Y and Toyama F M 1994 *Phys. Rev.* E **49** 4497

[8] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 *Rev. Mod. Phys.* **71** 463

[9] Sulem C and Sulem P L 1999 *The Nonlinear Schrödinger Equation* (New York: Springer)

[10] Theocharis G *et al* 2003 *Phys. Rev.* A **67** 063610

[11] Bilangér N and Bélanger P-A 1996 *Opt. Commun.* **124** 301

[12] Sakuma T *et al* 1984 *Phys. Rev.* B **28** 880

[13] Hasegawa A and Kodama Y 1981 *Proc. IEEE* **69** 1145
Hasegawa A and Kodama Y 1992 *Opt. Lett.* **7** 285